Complete Solution of the Extended EOQ Repair and Waste Disposal Model with Switching Costs

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Discussion Paper No. 376

November 2015

ISSN 1860 0921

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Abstract: The EOQ repair and waste disposal problem was studied first by Richter, 1997. A first shop is providing a homogeneous product used by a second shop at a constant demand rate. The first shop is manufacturing new products and it is also repairing products used by a second shop, which are then regarded as being as good as new. The products are employed by a second shop and collected there according to a repair rate. The other products are immediately disposed of as waste. At the end of some period of time, the collected products are brought back to the first shop and will be stored as long as necessary and then repaired. If the repaired products are finished, the manufacturing process starts to cover the remaining demand for the time interval. The model was extended by Saadany and Jaber, 2008 to the problem of minimizing the total cost of production, remanufacturing and inventory while incorporating additional switching costs. The switching cost is incurred when the process shifts from repair to production and from production to repair. However, in their paper the authors did not provide a complete solution to this complex problem. We provide the solution in this paper.

Keywords: EOQ model, Production/recovery, Reuse, Waste disposal, Switching cost

*The work is supported by Saint Petersburg State University grant 0.50.2097.2013.

1 Introduction

In recent years, reverse logistics has been receiving increasing attention from academia and industry.

There is increasing recognition that careful management can bring both environmental protection and lower costs; environmental and economic considerations have led to manufacturers taking their products back at the end of their lifetime. As a result, the reverse logistics process is now considered to be a basis for generating real economic value and to provide support for environmental concerns.

Rogers and Tibben-Lembke, 1998 [25] defined reverse logistics as the process of planning, implementing and controlling the efficient and cost effective flow of raw materials, in-process inventory, finished goods and related information from the point of consumption to the point of origin for the purpose of recapturing value or proper disposal.

The integration of forward and reverse supply chains resulted in the origination of the concept of a closed-loop supply chain. The whole chain can be designed in such a way that it can service both forward and reverse processes efficiently.

Akçalı and Çetinkaya, 2011 [1] published the most recent review of quantitative modelling for inventory and production planning in a closed-loop supply chain.

Inventory models are divided according to modelling demand and return processes into two main categories: deterministic and stochastic. The subject of this paper is deterministic inventory models with constant demand and return. The economic order quantity model (EOQ model), which was derived by Ford W. Harris in 1913, became the basis for many reverse logistics models because of its simplicity and intelligibility. Andriolo et al., 2014 [2] provided a most detailed review in their work on the EOQ problem. Shrady, 1967 [29] was the first to apply the EOQ model to reverse logistics processes. He introduced an EOQ model with instantaneous production and repair rates. A closed-form solution was developed. In his work an efficient policy $P(m, 1)$ was established, which means that within each remanufacturing cycle a number m of remanufacturing batches of equal size are followed by exactly one manufacturing batch.

This work was extended by Nahmias and Rivera, 1979 [19] and Mabini et al., 1992 [18] extended Shrady's model to the multi-item case. Koh et al., 2002 [11] analysed a model similar to that of Shrady, 1967 [29], but with some differences. They considered two types of policies,$P(m, 1)$ and $P(1, n)$, under a limited repair capacity, where n is the number of manufacturing batches. They examined the cases of a smaller and a larger recovery rate compared to the demand rate.

Teunter, 2001 [31] generalized the results of Schrady by examining different structures of the remanufacturing cycle. He considered different types of policies by placing the n manufacturing batches and m recovery batches in different orders. He concluded that the policy $P(m, n), m > 1, n > 1$ will never be optimal if the above-mentioned m and n are simultaneously larger than one, and that only the two policies $P(1, n)$ and $P(m, 1)$ are relevant.

Choi et al., 2007 [4] generalized the $P(m,n)$ policy of Teunter by considering the ordered sequence of manufacturing and remanufacturing batches within the cycle as decision variables. Through sensitivity analysis they found that only 0.2% out of the 8,100,000 tested instances of the model have an optimal solution with both m and n greater than one. Liu et al., 2009 [17] generated and solved 60,000 instances and found that only 0.19% of them have an optimal solution in $P(m,n)$ with both m and n greater than one. Konstantaras and Papachristos, 2008 [14] extended Teunters approach and found the exact solutions for the optimal numbers m and n.

In the literature two different types of problems are considered. Some authors have searched for an optimal policy $P(m,n)$ that involves determining the optimal number of manufacturing and remanufacturing batches (we call this problem "ONB") for given recovery or waste disposal rates β or α. Others have tried to go further by determining the optimal recovery or waste disposal (we call this problem "OWDR").

Richter was the author of a series of papers where he considered an EOQ model with respect to the waste disposal problem. Richter, 1996 [21] proposed an EOQ model that differed from that of Shrady, who assumed a continuous flow of used products to the manufacturer. Richter, 1996 [21] assumed a system of two shops: the first shop provided a product used by a second shop; the first shop manufactures new products and repairs (in contemporary terms—remanufactures) products already used by the second shop and collected there according to some rate; other products are disposed of according to a disposal rate. At the end of a certain time interval the collected items are brought back to the first shop. Richter, 1997 [23] examined the optimal inventory holding policy if the waste disposal (return) rate is a decision variable. The result of this study was that the optimal policy has an extremal property: either reuse all items without disposal or dispose of all items and produce new products; that is, the policy of the type $P(m,n)$ with $m > 1$ and $n > 1$ is never optimal. He also derived a closed-form for the optimal policy parameters. This analysis of the repair and waste disposal model was continued in the papers by Richter and Dobos, 1999 [24] and Dobos and Richter, 2000 [5].

Dobos and Richter, 2003 [6] and Dobos and Richter, 2004 [7] studied a production/recycling system with constant demand that is satisfied by non-instantaneous production and recycling. They concluded that it is optimal either to produce or to recycle all items that are brought back. Dobos and Richter, 2006 [8] extended their previous work by considering the quality of the returned items.

Saadany and Jaber 2010 [27] argued that such a pure policy of no waste disposal is technologically infeasible and suggested the introduction of a demand function that depends on two decision variables: purchasing price and acceptance quality level.

Saadany et al., 2012 [28] regarded the assumption that an item can be recovered indefinitely as unrealistic: material degrades in the process of recycling and loses some of its mass and quality, thereby making the option of multiple recovery somewhat infeasible. Saadany et al., 2012 [28] developed a model where an item can be recovered only a finite number of times.

Some authors extended the above-mentioned models to take account of various assumptions. One option is to allow for backorders, where some customers are compensated for having to wait for their delayed orders by either a reduction in price or some other form of discount, which is a cost incurred by the supplying firm. This results in a backorder cost. Konstantaras and Papachristos, 2006 [13] extended the work of Richter, 1996 [21] by allowing for backorders in remanufacturing and production while keeping the other assumptions the same. Saadany and Jaber, 2009 [10] extended the work of Richter, 1996 [21] by assuming that demand for manufactured items is different from that for remanufactured (repaired) items. This assumption results in lost sales situations where there are stock-out periods for manufactured and remanufactured items; that is, demand for newly manufactured items is lost during remanufacturing cycles and vice versa. In the study of Konstantaras et al., 2010 [16], which extended the work of Koh et al., 2002 [11], a combined inspection and sorting process is introduced with a fixed setup cost and unit variable costs. This study assumes that remanufactured and newly purchased products are sold in a primary market whereas refurbished units are sold in a secondary market. Konstantaras and Scouri, 2010 [15] considered two models: one with no shortages and the other with shortages. Both models are considered for the case of variable setup numbers of equal sized batches for the production and remanufacturing processes. For these two models, sufficient conditions for the optimal type of policy, referring to the parameters of the models, are proposed. Hasanov et al., 2012 [9] extended the work of Jaber and Saadany, 2009 [10] for the full-backorder and partial-backorder cases, where recovered items (remanufactured or repaired) are perceived by customers to be of lower quality; that is, not as good as new items.

Pishchulov et al., 2014 [20] studied a closed-loop supply chain in which a single purchaser orders a particular product from a single vendor and sells it on the market. A certain fraction of used items are returned to the purchaser from the market. The latter is responsible for collecting and returning them to the vendor. In addition to manufacturing new items, the vendor is able to remanufacture the returns into items that are as good as new and are subsequently used to meet the demand from the market. The questions addressed by this study pertain to the optimal centralized control of this closed-loop supply chain, the individually optimal policies of its members and the coordination within this supply chain under a decentralized control.

However, for some of the above-mentioned models, so far no complete solutions have been presented. In the paper of Saadany and Jaber, 2008 [26] the extended EOQ production, repair and waste disposal model of Richter, 1996 [21] was modified to show that ignoring the first time interval results in an unnecessary residual inventory and consequently an over estimation of the holding costs. They also introduced switching costs in order to take into account production losses, deterioration in quality or additional labour. When shifting from producing (performing) one product (job) to another in the same facility, the facility may incur additional costs, referred to as switching costs, when alternating between production and repair runs. The special case of even numbers m

and n was studied and conditions were provided to decide which of two policies $P(m,n)$ and $P(\frac{m}{2},\frac{n}{2})$ is preferable, but a general optimal policy for the problem was not presented. In our study we will provide a general optimal solution for the model.

Our paper is organized in the following way: in the second section, the assumptions and notations are presented; in the third section, the extended EOQ production, repair and waste disposal model, with switching costs, is formulated and analysed; in the fourth section, the ONB problem is studied, an exact optimal policy is derived and some numerical analysis is conducted; in the fifth section, the impact of the waste disposal rate to the numbers of batches is considered; the sixth section addresses the OWDR problem and the seventh section contains our conclusions.

2 Assumptions and notations

2.1 Assumptions

This paper assumes: (1) infinite manufacturing and recovery rates; (2) repaired items are as good as new; (3) demand is known, constant and independent; (4) the lead time is zero; (5) a single product case; (6) no shortages are allowed; (7) unlimited storage capacity is available; and (8) an infinite planning horizon.

2.2 Notations

T – length of a manufacturing and repairing time interval (units of time), where $T > 0$

T_1 – length of the first manufacturing time interval (units of time), where $T_1 < T$ and $T_1 > 0$

n – number of newly manufactured batches in an interval of length T

m – number of repaired batches in an interval of length T

d – demand rate (units per unit of time)

h – holding cost per unit per unit of time for shop 1

u – holding cost per unit per unit of time for shop 2

α – waste disposal rate, where $0 < \alpha < 1$

β – repair rate of used items, where $\alpha + \beta = 1$ and $0 < \beta < 1$

x – batch size for interval T, which includes n newly manufactured and m repaired batches; $x = dT$

r – repair setup cost per batch

s – manufacturing setup cost per batch

r_1 – setup and switching costs of the first repair run

s_1 – setup and switching costs of the first production run, denoted by $r_1 = r + \textit{switching cost}$ from production to repair, and $s_1 = s + \textit{switching cost}$ from repair to production.

3 Formulation of the model and its analysis

Richter, 1996 [21] introduced an EOQ repair and waste disposal model. A first shop is providing a homogeneous product used by a second shop at a constant demand rate of d items per time unit. The first shop is manufacturing new products and it is also repairing products used by a second shop, which are then regarded as being as good as new. The products are employed by a second shop and collected there according to a repair rate β. The other products are immediately disposed of as waste according to the waste disposal rate $\alpha = 1-\beta$. At the end of some period of time $[0, T]$, the collected products are brought back to the first shop and will be stored as long as necessary and then repaired. If the repaired products are finished, the manufacturing process starts to cover the remaining demand for the time interval. The switching cost is incurred when the process shifts from repair to production and from production to repair. In the study of Saadany and Jaber, 2008 [26], the holding cost expression in Richter's model was modified because of the effect of the first time interval (see Fig.1). This helps to reduce the total inventories of all the subsequent time intervals.

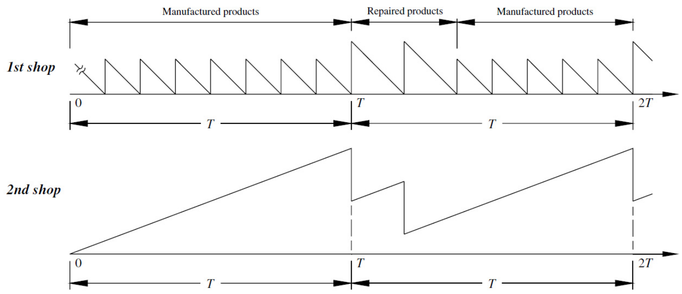

Figure 1: The modified behavior of inventory in the 1st and 2nd shops

According to Saadany and Jaber, 2008 [26], the modified cost function in the model of Richter, 1996 [23] with switching costs is equal to

$$K_2(x, m, n, \alpha) = ((m-1)r + r_1 + (n-1)s + s_1) + \\ + \frac{h}{2d}\left(\frac{\alpha^2 x^2}{n} + \frac{\beta^2 x^2}{m}\right) + \frac{u\beta T x}{2} - \frac{u\beta^2 x^2 (m-1)}{2dm}.$$

The modified cost per time unit function is obtained by dividing by T

$$K(x, m, n, \alpha) = \frac{K_2(x, m, n, \alpha)}{T} = \frac{d}{x}((m-1)r + r_1 + (n-1)s + s_1) + \\ + \frac{x}{2}\left[h\left(\frac{\alpha^2}{n} + \frac{\beta^2}{m}\right) + u\beta - \frac{u\beta^2(m-1)}{m}\right], \quad (1)$$

where $x = dT$. The function (1) is convex and differentiable in x, therefore there is a unique minimum point

$$x(m,n,\alpha) = \sqrt{\frac{2d((m-1)r + r_1 + (n-1)s + s_1)}{h(\frac{\alpha^2}{n} + \frac{\beta^2}{m}) + u\beta - u\beta^2(\frac{m-1}{m})}}. \tag{2}$$

The minimum cost per time unit for given values m, n, α is obtained by substituting (2) into (1):

$$K(m,n,\alpha) = \\ = \sqrt{2d(mr + ns + r_1 + s_1 - r - s)\left(h(\frac{\alpha^2}{n} + \frac{\beta^2}{m}) + u\beta - \frac{u\beta^2(m-1)}{m}\right)}. \tag{3}$$

4 Determining the optimal policy for the generalized EOQ waste and disposal model (ONB)

To determine the optimal policy means to find the optimal numbers m and n for the minimum cost found in the previous section (3) (the ONB problem). In this section α will be a constant and not a variable. Therefore, the function (3) will be denoted just by $K(m,n)$. The problem of determining the optimal batch numbers takes the following form as a nonlinear integer optimization problem (4)

$$\begin{aligned} &\min_{(m,n)} K(m,n), \\ &m, n \in \{1, 2, \dots\}. \end{aligned} \tag{4}$$

The determination of optimal values for m, n and later also α, constitutes the problem of our paper and of other studies as well.

In order to derive explicit expressions for the optimal values in problem (4) let us first introduce the notations

$$\begin{aligned} &W = s_1 - s + r_1 - r, \;\; a_1 = \beta u - \beta^2 u = \alpha\beta u, \\ &a_2 = \beta^2(h+u), \;\; a_3 = \alpha^2 h, \\ &S = s, \;\; R = r. \end{aligned} \tag{5}$$

The parameter W can be treated as the "total net" switching cost. One can see that all parameters W, S, R, a_1, a_2, a_3 are positive. Then the function (3) can be expressed by

$$K(m,n) = \sqrt{2d(W + mR + nS)(a_1 + \frac{a_2}{m} + \frac{a_3}{n})}. \tag{6}$$

Let the radicand of the root (6) be denoted by

$$L(m,n) = (W + mR + nS)(a_1 + \frac{a_2}{m} + \frac{a_3}{n}). \tag{7}$$

Instead of solving the problem (4) the function (7) can be minimized $m \geq 1, n \geq 1$, i.e., the following two-dimensional nonlinear integer optimization problem is relevant:

$$\begin{aligned} &\min_{(m,n)} L(m,n) = \min_{(m,n)} (W + mR + nS)(a_1 + \frac{a_2}{m} + \frac{a_3}{n}), \\ &m, n \in \{1, 2, \dots\}. \end{aligned} \tag{8}$$

First, let us consider the following continuous auxiliary problem:

$$\begin{aligned} &\min_{(m,n)} L(m,n) = \min_{(m,n)} (W + mR + nS)(a_1 + \frac{a_2}{m} + \frac{a_3}{n}), \\ &m, n \in R,\ m \geq 1,\ n \geq 1. \end{aligned} \tag{9}$$

By analyzing the first partial derivatives

$$\begin{aligned} \frac{\partial L(m,n)}{\partial m} &= R(a_1 + \frac{a_3}{n}) - \frac{a_2}{m^2}(W + nS), \\ \frac{\partial L(m,n)}{\partial n} &= S(a_1 + \frac{a_2}{m}) - \frac{a_3}{n^2}(W + mR), \end{aligned} \tag{10}$$

we can formulate the following lemma:

Lemma 1. *If $m > 0, n > 0$, there are two curves of local minima (7) with respect to m:*

$$N(m) = \sqrt{\frac{a_3 m (W + mR)}{S(a_1 m + a_2)}}, \tag{11}$$

with respect to n :

$$M(n) = \sqrt{\frac{a_2 n (W + nS)}{R(a_1 n + a_3)}}, \tag{12}$$

and the point of local minimum:

$$(m^*, n^*) = \left(\sqrt{\frac{W a_2}{R a_1}}, \sqrt{\frac{W a_3}{S a_1}} \right). \tag{13}$$

For the proof of Lemma 1 see the Appendix A.

Let us denote the radicands of the expressions (13) by

$$\begin{aligned} A = \frac{W a_2}{R a_1} &= \frac{(s_1 - s + r_1 - r)\beta(h + u)}{r \alpha u}, \\ B = \frac{W a_3}{S a_1} &= \frac{(s_1 - s + r_1 - r)\alpha h}{s \beta u}, \end{aligned} \tag{14}$$

and the value of $M(n)$ (12), if $n = 1$, and $N(m)$ (11) if $m = 1$ by C and D:

$$\begin{aligned} C = M(1) &= \frac{a_2(S+W)}{R(a_1+a_3)}, \\ D = N(1) &= \frac{a_3(W+R)}{S(a_1+a_2)}. \end{aligned} \tag{15}$$

Then the optimal solution for continuous problem (9) is provided by the following theorem (The detailed proof is contained in the Appendix A.)

Theorem 1. *The optimal solution to the problem (9) has the following structure depending on the value of the parameters A, B, C, D:*

1. *If $A \geq 1,\ B \geq 1,$ then $m = \sqrt{A},\ n = \sqrt{B},$*
   $L(\sqrt{A}, \sqrt{B}) = L_1 = (\sqrt{Wa_1} + \sqrt{Ra_2} + \sqrt{Sa_3})^2$
2. *If $A < 1$ or $B < 1$ and $C \geq 1,\ D < 1,$ then $m = \sqrt{C},\ n = 1,$*
   $L(\sqrt{C}, 1) = L_2 = (\sqrt{(W+S)(a_1+a_3)} + \sqrt{Ra_2})^2$
3. *If $A < 1$ or $B < 1$ and $C < 1,\ D \geq 1,$ then $m = 1,\ n = \sqrt{D},$*
   $L(1, \sqrt{D}) = L_3 = (\sqrt{(W+R)(a_1+a_2)} + \sqrt{Sa_3})^2$
4. *If $A < 1$ or $B < 1$ and $C < 1,\ D < 1,$ then $m = 1\ n = 1,$*
   $L(1,1) = L_4 = (W+R+S)(a_1+a_2+a_3).$

By applying this result the optimal solution to the original problem (8) can be easily derived:

Theorem 2. *The optimal solution to the problem (8) has the following structure depending on the value of the parameters A, B, C, D:*

1. *If $A \geq 1$ and $B \geq 1,$ then*

$$\begin{aligned} (m,n) &= \\ &= \arg\min\{L([\sqrt{A}], [\sqrt{B}]), L([\sqrt{A}]+1, [\sqrt{B}]), \\ &\qquad L([\sqrt{A}], [\sqrt{B}]+1), L([\sqrt{A}]+1, [\sqrt{B}]+1)\} \end{aligned}$$

2. *If $A < 1$ or $B < 1$ and $C \geq 1,\ D < 1,$ then*

$$(m,n) = \arg\min\{L([\sqrt{C}], 1), L([\sqrt{C}]+1, 1)\},$$

3. *If $A < 1$ or $B < 1$ and $C < 1,\ D \geq 1,$ then*

$$(m,n) = \arg\min\{L(1, [\sqrt{D}]), L(1, [\sqrt{D}]+1)\},$$

4. *If $A < 1$ or $B < 1$ and $C < 1,\ D < 1,$ then*

$$(m,n) = (1,1),$$

where $[\dots]$ denotes the integer (or floor) part of a number.

The proof of this theorem follows from the quasi convexity of the function $L(m,n)$.

Numerical analysis The input parameters for numerical analysis are represented in the Table 1. Each of the model parameters has been set to vary in a range, which are represented in the Table 1.

Table 1: The input parameters for the numerical analysis.

	d	α	h	u	r	s	$r_1 + s_1$
Max	10000	0	1	1	1	1	1
Min	10000	1	50	50	500	500	1000

The minimum and maximum values of parameters s and h were chosen with respect to

$$20 \leq \sqrt{\frac{2ds}{h}} \leq 10000,$$

where $x = \sqrt{\frac{2ds}{h}}$ is the classical EOQ value for the non-remanufacturing case. The sets of parameters $(h, u, r, s, r_1 + s_1)$ for 10,000 instances were randomly generated. When generating u, h and $r_1 + s_1$, the constraints $h > u$ and $r_1 + s_1 > r + s$ were respected. According to our study, the policy $P(m, n)$ with $m > 1, n > 1$ is optimal for 2304 instances out of 10,000; some more results are displayed in Table 2.

Table 2: Results of the numerical analysis.

$P(m,n), m > 1, n > 1$	$P(1,n)$	$P(m,1)$	$P(1,1)$
2304	2808	3756	1132

5 The impact of disposal and return rates

Consider now the impact of disposal and return rates on the numbers of batches. In other words, let us determine for which values of α the four different structures of the optimal solution of theorems 1 and 2 appear.

Recall W will be the switching costs:

$$W = s_1 + r_1 - s - r.$$

First let us rewrite the formulas (14) and (15) by

$$\begin{aligned}
A(\alpha) &= \frac{W(h+u)}{ru}\frac{1-\alpha}{\alpha},\\
B(\alpha) &= \frac{Wh}{su}\frac{\alpha}{1-\alpha},\\
C(\alpha) &= \frac{(W+s)(h+u)}{r}\frac{(1-\alpha)^2}{\alpha(u+\alpha(h-u))},\\
D(\alpha) &= \frac{(W+r)h}{s}\frac{\alpha^2}{(1-\alpha)(u+h-\alpha h)}.
\end{aligned}$$

Now let us formulate four properties of these functions. Some auxiliary propositions are formulated. (The proofs are contained in Appendix B.)

Property 1. *The functions $A(\alpha), C(\alpha)$ are positive and decreasing if $\alpha \in (0,1)$ and the functions $B(\alpha), D(\alpha)$ are positive and increasing if $\alpha \in (0,1)$.*

Property 2. *Each of the equations $A(\alpha) = 1,\ B(\alpha) = 1,\ C(\alpha) = 1,\ D(\alpha) = 1$ has a unique solution $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, correspondingly, if $\alpha \in (0,1)$:*

$$\begin{aligned}
\alpha_1 &= \frac{W(h+u)}{ru+W(h+u)},\\
\alpha_2 &= \frac{su}{su+Wh},\\
\alpha_3 &= \begin{cases} \frac{1}{2}\frac{2(W+s)(h+u)+ru-\sqrt{4hr(W+s)(h+u)+r^2u^2}}{(W+s)(h+u)-r(h-u)}, & (W+s)(h+u) \neq r(h-u) \\ \frac{h-u}{2h+u}, \quad (W+s)(h+u)-r(h-u)=0 & \end{cases}\\
\alpha_4 &= \begin{cases} \frac{1}{2}\frac{-s(u+2h)+\sqrt{4hs(W+r)(h+u)+u^2s^2}}{h(W+r-s)}, & W+r-s \neq 0 \\ \frac{u+h}{u+2h}, \quad W+r-s=0 & \end{cases}.
\end{aligned}$$

Property 3. *The functions $A(\alpha)$ and $C(\alpha)$ have two common points, if $\alpha \in (0,1]$: $(1,0)$ and $(\alpha_2, \frac{W^2h(h+u)}{rsu^2})$; moreover: $A(\alpha) > C(\alpha)$, if $\alpha \in (\alpha_2, 1)$, $A(\alpha) < C(\alpha)$, if $\alpha \in (0, \alpha_2)$. The functions $B(\alpha)$ and $D(\alpha)$ have two common points, if $\alpha \in [0,1)$: $(0,0)$ and $(\alpha_1, \frac{W^2h(h+u)}{rsu^2})$; moreover: $B(\alpha) > D(\alpha)$, if $\alpha \in (0, \alpha_1)$, $B(\alpha) < D(\alpha)$ if $\alpha \in (\alpha_1, 1)$.*

Property 4. *The equations $A(\alpha) = B(\alpha)$ and $C(\alpha) = D(\alpha)$ have unique solutions α^* and α^{**}, correspondingly. Moreover, if $C(\alpha^{**}) > 1$ then $A(\alpha^*) > 1$, if $C(\alpha^{**}) < 1$ then $A(\alpha^*) < 1$, if $C(\alpha^{**}) = 1$ then $A(\alpha^*) = 1$ and $\alpha^* = \alpha^{**}$, where*

$$\begin{aligned}
\alpha^* &= \frac{\sqrt{\frac{(h+u)s}{ru}}}{1+\sqrt{\frac{(h+u)s}{ru}}}\\
A(\alpha^*) &= B(\alpha^*) = \frac{W}{u}\sqrt{\frac{h(h+u)}{rs}}
\end{aligned} \tag{16}$$

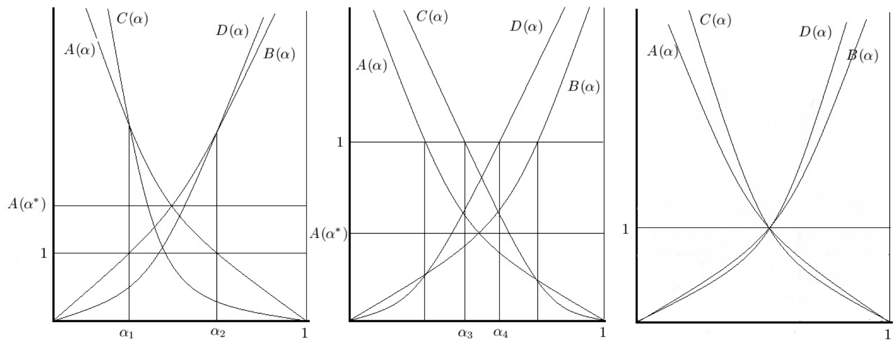

Figure 2: Behaviour of the functions $A(\alpha), B(\alpha), C(\alpha), D(\alpha)$

Taking into account Statement 4, there are three different types of positional relationships of curves $A(\alpha), B(\alpha), C(\alpha), D(\alpha)$ (compare Fig. 2):

1. The intersection point of $A(\alpha)$ and $B(\alpha)$ is lower than 1: $A(\alpha^*) < 1$
2. The intersection point of $A(\alpha)$ and $B(\alpha)$ is larger then 1: $A(\alpha^*) > 1$
3. All four curves have one intersection point: $A(\alpha^*) = B(\alpha^*) = C(\alpha^*) = D(\alpha^*) = 1$.

The four different structures for the objective function appear due to the four distinct relations between the values A, B, C and D (see Theorem 1):

1. If $A \geq 1,\ B \geq 1$, then $m \geq 1,\ n \geq 1$;
2. If $A < 1$ or $B < 1$ and $C \geq 1,\ D < 1$, then $m \geq 1,\ n = 1$;
3. If $A < 1$ or $B < 1$ and $C < 1,\ D \geq 1$, then $m = 1,\ n \geq 1$;
4. If $A < 1$ or $B < 1$ and $C < 1,\ D < 1$, then $m = 1\ n = 1$.

It is obvious that the second and third cases will appear independently of the positional relationships of curves $A(\alpha), B(\alpha), C(\alpha), D(\alpha)$. The appearance of the first and fourth cases depends on the intersections of the curves. The condition $A(\alpha^*) > 1$, can be rewritten in the following form:

$$W > u\sqrt{\frac{rs}{h(h+u)}}. \tag{17}$$

It can be seen that the condition (17) is equivalent to the relation

$$\alpha_1 > \alpha_2.$$

Denote the right hand side of inequality (17) by:

$$W^* = u\sqrt{\frac{rs}{h(h+u)}}.$$

If $W > W^*$ there exists an interval when simultaneously $A > 1$ and $B > 1$: (α_1, α_2), this is the first case of theorem 1. Therefore,

$$\begin{aligned} m \geq 1, n = 1, &\quad \alpha \in (0, \alpha_1) \\ m \geq 1, n \geq 1, &\quad \alpha \in [\alpha_1, \alpha_2] \\ m = 1, n \geq 1, &\quad \alpha \in (\alpha_2, 1) \end{aligned}$$

On the other hand, if $W < W^*$, we obtain that at any $\alpha \in (0,1)$ either $A(\alpha)$ or $B(\alpha)$ is less then one, here we have second, third and fourth cases of theorem 1:

$$\begin{aligned} m \geq 1, n = 1, &\quad \alpha \in (0, \alpha_3] \\ m = 1, n = 1, &\quad \alpha \in (\alpha_3, \alpha_4) \\ m = 1, n \geq 1, &\quad \alpha \in [\alpha_4, 1) \end{aligned}$$

In the third situation, if $W = W^*$, when the four curves have a common unique intersection, $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha^* = \alpha^{**}$, we have:

$$\begin{aligned} m \geq 1, n = 1, &\quad \alpha \in (0, \alpha^*] \\ m = 1, n \geq 1, &\quad \alpha \in [\alpha^*, 1) \end{aligned}$$

Numerical example 1 Consider a case with the parameters $d = 10000,\ h = 5,\ u = 2,\ r = 30,\ s = 90,\ r_1 = 50,\ s_1 = 150$. Consider different values of parameter $\alpha \in (0,1)$. Switching Costs: $W = 80$.

It can be easily calculated that $A(\alpha^*) = 4,55$ and $W^* = 17,57$. Here we have the appearance of the first case of Theorem 1. We find that: $\alpha_1 = 0,31,\ \alpha_2 = 0,903$. Consider also the case when all parameters are the same but there is no switching cost in consideration, i.e., $W = 0$. Denote by $\bar{K}(m,n,\alpha)$ the corresponding cost. For the results see Fig. 3.

Numerical example 2 Consider a case with parameters: $d = 10000,\ h = 5,\ u = 2,\ r = 30,\ s = 90,\ r_1 = 32,\ s_1 = 93$. Consider different values of parameter $\alpha \in (0,1)$. Switching Costs $W = 5$. $A(\alpha^*) = 0,29$. $W^* = 17,57$. We have that $W^* > W$ and then find $\alpha_3 = 0,657,\ \alpha_4 = 0,713$. The results are displayed in Fig. 4.

6 The optimal waste disposal rate (problem OWDR)

In this section the problem OWDR is considered. Let $\alpha \in [0,1]$. Note that on the one hand, in some situations no remanufacturing is suitable. In this case

$\alpha = 1,\ m = 0,\ n = 1$, and the model reduces to the classical EOQ:

$$K^{EOQ}_{\alpha=1}(x) = \frac{ds}{x} + \frac{hx}{2}$$

with the optimal cost equal to $\sqrt{2dsh}$. On the other hand, if all products are remanufactured, then $\alpha = 0,\ m = 1,\ n = 1$ and the model would be:

$$K^{EOQ}_{\alpha=0}(x) = \frac{dr}{x} + \frac{(h+u)x}{2}$$

and the optimal cost would be equal to $\sqrt{2dr(u+h)}$.

Recall that for all other α the function $K(m,n)$ was defined as (6): $K(m,n) = \sqrt{2dL(m,n)}$.

Let us denote

$$\begin{aligned}
K_1 &= \sqrt{2dL_1} = \sqrt{2d}(\sqrt{Wa_1} + \sqrt{Ra_2} + \sqrt{Sa_3}),\\
K_2 &= \sqrt{2dL_2} = \sqrt{2d}(\sqrt{(W+S)(a_1+a_3)} + \sqrt{Ra_2}),\\
K_3 &= \sqrt{2dL_3} = \sqrt{2d}(\sqrt{(W+R)(a_1+a_2)} + \sqrt{Sa_3}),\\
K_4 &= \sqrt{2dL_4} = \sqrt{2d(W+R+S)(a_1+a_2+a_3)}.
\end{aligned}$$

It can be easily proved that

$$\begin{aligned}
K_4 \geq K_2 \geq K_1,\\
K_4 \geq K_3 \geq K_1.
\end{aligned}$$

Substituting the formulas (5) for the initial parameters gives

$$K_1 = \sqrt{2d}(\sqrt{W\alpha\beta u} + \alpha\sqrt{sh} + \beta\sqrt{r(h+u)}) \geq$$
$$\sqrt{2d}(\sqrt{W\alpha\beta u} + \min\{\sqrt{sh}, \sqrt{r(h+u)}\}) \geq \min\{\sqrt{2dsh}, \sqrt{2dr(h+u)}\}.$$

We obtained that if $\alpha \in [0,1]$, the optimal strategy for minimizing the total costs will be $\alpha = 1$ and $\alpha = 0$ with the costs $\sqrt{2dsh}$ or $\sqrt{2dr(u+h)}$, correspondingly. The case $\alpha \in [\alpha_{min}, \alpha_{max}]$, $0 < \alpha_{min} \leq \alpha \leq \alpha_{max} < 1$ will be studied in the future.

7 Summary and conclusions

In this paper we analysed the extended EOQ repair and waste disposal model with switching costs. Two problems were considered: ONB and OWDR.

For the OWDR problem, we proved that the optimal strategy will be to dispose of all used products or to remanufacture them all. This result agrees with other results for similar problems.

We found the optimal policy $P(m,n)$ for the ONB problem; it can have a different structure depending on the value of the parameters A, B, C, D. The optimal policy (m,n) depends on the disposal rate α, ceteris paribus; in other words, the higher the α and the higher the m, the lower the n. The impact of the switching cost becomes apparent for sufficiently high values. In this case the optimal numbers (m, n) can both be greater than one. This was illustrated by the examples. To our knowledge, the case of having both optimal numbers m and n greater then one, if m remanufacturing batches are followed by the sequence of n manufacturing batches or vice versa, has not been previously mentioned in the literature. Choi et al. [4] found solutions with n and m both greater than one, but they had placed the n manufacturing batches and m recovery batches in different orders and considered the ordered sequence of manufacturing and remanufacturing batches within the cycle as decision variables. They found that only 0.2% of the 8,100,000 tested problems had an optimal solution with both m and n greater then one. In this study we conducted a numerical analysis for an EOQ repair and waste disposal model with switching costs and we found that the optimal m and n are both greater then one in about 23% of 10,000 different sets of parameters.

Acknowledgements

First author wish to thank Saint Petersburg State University for supporting this research.

Appendix A

Lemma 1. *If $m > 0, n > 0$, there are two curves of local minima (9) with respect to m:*

$$N(m) = \sqrt{\frac{a_3 m(W + mR)}{S(a_1 m + a_2)}}, \tag{18}$$

with respect to n :

$$M(n) = \sqrt{\frac{a_2 n(W + nS)}{R(a_1 n + a_3)}}, \tag{19}$$

and the point of local minimum:

$$(m^*, n^*) = \left(\sqrt{\frac{Wa_2}{Ra_1}}, \sqrt{\frac{Wa_3}{Sa_1}}\right). \tag{20}$$

Proof. It follows from (10) that

$$\frac{\partial L(m, n)}{\partial m} < 0 \Leftrightarrow R(a_1 + \frac{a_3}{n}) - \frac{a_2}{m^2}(W + nS) < 0 \Leftrightarrow m < M(n).$$

In other words, the function $L(m, n)$ decreases in m, if $m < M(n)$ and increases in m, if $m > M(n)$.

$$\frac{\partial^2 L}{\partial m^2} = \frac{2a_2(W + nS)}{m^3} > 0,$$

if $m > 0, n > 0$. This means that $L(m,n)$ is convex in m, therefore $M(n)$ is the curve of local minimum in m.

Substituting the expression for m (12) into (9) leads to

$$L(M(n),n) = 2\sqrt{\frac{a_2R(Sn+W)(a_1n+a_3)}{n}} + Sa_1n + \frac{Wa_3}{n} + Ra_2 + Sa_3.$$

By differentiating $L(M(n),n)$ with respect to n we receive

$$L'(M(n),n) = \left(Sa_1 - \frac{Wa_3}{n^2}\right)\left(1 + \sqrt{\frac{Ra_2n}{(Sn+W)(a_1n+a_3)}}\right)$$

and as the result

$$m^* = \sqrt{\frac{Wa_2}{Ra_1}}, \;\; n^* = N(m^*) = \sqrt{\frac{Wa_3}{Sa_1}}.$$

Since the Hessian

$$D(m^*,n^*) = \left|\begin{matrix} \frac{\partial^2 L}{\partial m^2} & \frac{\partial^2 L}{\partial m \partial n} \\ \frac{\partial^2 L}{\partial n \partial m} & \frac{\partial^2 L}{\partial n^2} \end{matrix}\right|_{(m^*,n^*)} = \Bigg(\frac{4a_2a_3W(W+nS+mR)}{m^3n^3} -$$
$$-\left(\frac{Ra_3}{n^2} - \frac{Sa_2}{m^2}\right)^2\Bigg)_{(m^*,n^*)} = \frac{4a_2a_3W(W+n^*S+m^*R)}{(m^*)^3(n^*)^3} \geq 0$$

is positively definite matrix, (20) is the point of local minimum of the function (7). □

Theorem 1. *The optimal solution to the problem (9) has the following structure depending on the value of the parameters A, B, C, D:*

1. *If $A \geq 1,\; B \geq 1$, then $m = \sqrt{A},\; n = \sqrt{B}$,*
   $L = L_1 = (\sqrt{Wa_1} + \sqrt{Ra_2} + \sqrt{Sa_3})^2$
2. *If $A < 1$ or $B < 1$ and $C \geq 1,\; D < 1$, then $m = \sqrt{C},\; n = 1$,*
   $L = L_2 = (\sqrt{(W+S)(a_1+a_3)} + \sqrt{Ra_2})^2$
3. *If $A < 1$ or $B < 1$ and $C < 1,\; D \geq 1$, then $m = 1,\; n = \sqrt{D}$,*
   $L = L_3 = (\sqrt{(W+R)(a_1+a_2)} + \sqrt{Sa_3})^2$
4. *If $A < 1$ or $B < 1$ and $C < 1,\; D < 1$, then $m = 1\; n = 1$,*
   $L = L_4 = (W+R+S)(a_1+a_2+a_3)$

Proof. The outline of the proof:

1. To find the optimal (m,n) using the Kuhn–Tucker conditions supposing $L(m,n)$ to be the convex function.
2. To prove that the function $L(m,n)$ is convex at least at the point (m^*,n^*).

3. To prove that the function $L(m,n)$ is quasi convex at $m>0, n>0$.
4. To prove that the (m,n) which satisfies the Kuhn–Tucker conditions is optimal using quasi convexity of $L(m,n)$ and the Arrow and Enthoven, 1961 [3] theorem.
5. To find the values L_1, L_2, L_3, L_4

1. Recall that

$$A = \frac{Wa_2}{Ra_1}$$
$$B = \frac{Wa_3}{Sa_1}$$
$$C = \frac{a_2(S+W)}{R(a_1+a_3)}$$
$$D = \frac{a_3(W+R)}{S(a_1+a_2)}.$$

Let function $L(m,n)$ be the convex function, then the Kuhn–Tucker conditions for problem (9) are as follows:

$$\begin{aligned}
&R(a_1+\frac{a_3}{n}) - \frac{a_2}{m^2}(W+nS) - \lambda_1 = 0\\
&S(a_1+\frac{a_2}{m}) - \frac{a_3}{n^2}(W+mR) - \lambda_2 = 0\\
&m-1 \geq 0\\
&n-1 \geq 0\\
&\lambda_1(m-1) = 0\\
&\lambda_2(n-1) = 0\\
&\lambda_1 \geq 0,\ \lambda_2 \geq 0.
\end{aligned} \tag{21}$$

Let us denote

$$\begin{aligned}
\lambda_1(m,n) &= R(a_1+\frac{a_3}{n}) - \frac{a_2}{m^2}(W+nS),\\
\lambda_2(m,n) &= S(a_1+\frac{a_2}{m}) - \frac{a_3}{n^2}(W+mR).
\end{aligned} \tag{22}$$

Recall that $a_2>0,\ a_3>0$. The condition $\lambda_1(1,1)>0$ is equivalent to the condition $C<1$ and similarly $\lambda_2(1,1)>0 \Leftrightarrow D<1$. If $\lambda_1(1,1)>0(\Leftrightarrow C<1)$ and $\lambda_2(1,1)>0(\Leftrightarrow D<1)$ then

$$\begin{cases}
m = 1\\
n = 1\\
\lambda_1 = \lambda_1(1,1) > 0\\
\lambda_2 = \lambda_2(1,1) > 0
\end{cases}$$

satisfy the Kuhn–Tucker conditions (21). If $\lambda_1(1,1) > 0(\Leftrightarrow C < 1)$ and $\lambda_2(1,1) \le 0(\Leftrightarrow D \ge 1)$ then consider

$$\lambda_1(1,\sqrt{D}) = (Ra_1 - Wa_2)(1 + \frac{S}{W+R}\sqrt{D}).$$

If $Ra_1 - Wa_2 > 0(\Leftrightarrow A < 1)$ then

$$\begin{cases} m = 1 \\ n = N(1) = \sqrt{D} \\ \lambda_1 = \lambda_1(1,\sqrt{D}) > 0 \\ \lambda_2 = 0 \end{cases}$$

satisfy the Kuhn–Tucker conditions (21). If $Ra_1 - Wa_2 < 0(\Leftrightarrow A > 1)$ then

$$\begin{cases} C < 1 \\ A > 1 \end{cases} \Leftrightarrow \begin{cases} \underbrace{(Wa_2 - Ra_1)}_{>0} + (Sa_2 - Ra_3) < 0 \\ A > 1 \end{cases} \Rightarrow$$

$$\begin{cases} Sa_2 - Ra_3 < 0 \\ A > 1 \end{cases} \Leftrightarrow \begin{cases} \frac{A}{B} < 1 \\ A > 1 \end{cases} \Rightarrow 1 < A < B \Rightarrow B > 1$$

which means that

$$\begin{cases} m = \sqrt{A} \\ n = \sqrt{B} \\ \lambda_1 = \lambda_1(\sqrt{A},\sqrt{B}) = 0 \\ \lambda_2 = \lambda_1(\sqrt{A},\sqrt{B}) = 0 \end{cases}$$

satisfy the Kuhn–Tucker conditions (21). In the same way, if If $\lambda_1(1,1) \le 0(\Leftrightarrow C \ge 1)$ and $\lambda_2(1,1) \le 0(\Leftrightarrow D \ge 1)$ then Let $Sa_2 - Ra_3 > 0$ then

$$\begin{cases} D \ge 1 \\ Sa_2 - Ra_3 > 0 \end{cases} \Leftrightarrow \begin{cases} (Wa_3 - Sa_1) - \underbrace{(Sa_2 - Ra_3)}_{>0} \ge 0 \\ A > B \end{cases} \Rightarrow$$

$$\begin{cases} Wa_3 - Sa_1 \ge 0 \\ A > B \end{cases} \Rightarrow \begin{cases} B \ge 1 \\ A > B \end{cases} \Rightarrow \begin{cases} A \ge 1 \\ B \ge 1 \end{cases}.$$

Let $Sa_2 - Ra_3 < 0$ then

$$\begin{cases} C \ge 1 \\ Sa_2 - Ra_3 < 0 \end{cases} \Leftrightarrow \begin{cases} (Wa_2 - Ra_1) + \underbrace{(Sa_2 - Ra_3)}_{<0} \ge 0 \\ A < B \end{cases} \Rightarrow$$

$$\begin{cases} Wa_2 - Ra_1 \ge 0 \\ A < B \end{cases} \Rightarrow \begin{cases} A \ge 1 \\ A < B \end{cases} \Rightarrow \begin{cases} A \ge 1 \\ B \ge 1 \end{cases}.$$

In any case

$$\begin{cases} m = \sqrt{A} \\ n = \sqrt{B} \\ \lambda_1 = \lambda_1(\sqrt{A}, \sqrt{B}) = 0 \\ \lambda_2 = \lambda_1(\sqrt{A}, \sqrt{B}) = 0 \end{cases}$$

satisfy the Kuhn–Tucker conditions (21). *We obtain that*

- *If* $C < 1, D < 1$ *then* $m = 1,\ n = 1$
- *If* $C < 1, D \geq 1$ *then*
  - *If* $A < 1$, *then* $m = 1, n = \sqrt{D}$
  - *If* $A \geq 1$, *then* $B \geq 1$ *and* $m = \sqrt{A}, n = \sqrt{B}$
- *If* $C \geq 1, D > 1$ *then*
  - *If* $B < 1$, *then* $m = \sqrt{C}, n = 1$
  - *If* $B \geq 1$, *then* $A \geq 1$ *and* $m = \sqrt{A}, = \sqrt{B}$
- *If* $C \geq 1, D \geq 1$ *then* $A \geq 1, B \geq 1$ *and* $m = \sqrt{A}, n = \sqrt{B}$.

2. Consider now concavity of the function $L(m, n)$. As a result we have:

$$\frac{\partial^2 L}{\partial m^2} = \frac{2a_2(W + nS)}{m^3} > 0,\ \frac{\partial^2 L}{\partial n^2} = \frac{2a_3(W + mR)}{n^3} > 0.$$

$$D(m, n) = \begin{vmatrix} \frac{\partial^2 L}{\partial m^2} & \frac{\partial^2 L}{\partial m \partial n} \\ \frac{\partial^2 L}{\partial n \partial m} & \frac{\partial^2 L}{\partial n^2} \end{vmatrix} = \frac{4a_2a_3W(W + nS + mR)}{m^3n^3} - \left(\frac{Ra_3}{n^2} - \frac{Sa_2}{m^2}\right)^2 \geq 0$$

at point $(m^*, n^*) = (\sqrt{\frac{Wa_2}{Ra_1}}, \sqrt{\frac{Wa_3}{Sa_1}})$ and at some epsilon neighborhood of this point. It means that $L(m, n)$ is concave at least at some epsilon neighborhood of (m^*, n^*).

3. Now we prove that $L(m, n)$ is quasi convex. Bordered Hessians are equal to

$$B_1(m, n) = \begin{vmatrix} 0 & \frac{\partial L}{\partial m} \\ \frac{\partial L}{\partial m} & \frac{\partial^2 L}{\partial m^2} \end{vmatrix} = -\left(\frac{\partial L}{\partial m}\right)^2 \leq 0$$

$$B_2(m, n) = \begin{vmatrix} 0 & \frac{\partial L}{\partial m} & \frac{\partial L}{\partial n} \\ \frac{\partial L}{\partial m} & \frac{\partial^2 L}{\partial m^2} & \frac{\partial^2 L}{\partial m \partial n} \\ \frac{\partial L}{\partial n} & \frac{\partial^2 L}{\partial n \partial m} & \frac{\partial^2 L}{\partial n^2} \end{vmatrix} = -\left(\frac{\partial L}{\partial m}\right)^2 \frac{2a_3(W + mR)}{n^3} -$$

$$-\left(\frac{\partial L}{\partial n}\right)^2 \frac{2a_2(W + nS)}{m^3} - 2\frac{\partial L}{\partial m}\frac{\partial L}{\partial n}\left(\frac{Ra_3}{n^2} + \frac{Sa_2}{m^2}\right) = -2(W + nS + mR) \cdot$$

$$\cdot \left(a_1\left(\frac{Ra_3}{n^2} - \frac{Sa_2}{m^2}\right)^2 + \frac{a_2}{m}\left(\frac{Sa_1}{m} - \frac{Wa_3}{mn^2}\right)^2 + \frac{a_3}{n}\left(\frac{Ra_1}{n} - \frac{Wa_2}{nm^2}\right)^2\right) < 0,$$

if $m \neq m^*,\ n \neq n^*$. But at (m^*, n^*) the function $L(m,n)$ is convex. This means that $L(m,n)$ is quasi convex at $m > 0, n > 0$.

4. And now we verify conditions from the following theorem: Theorem, Arrow and Enthoven, 1961[3]. *Let $f(x)$ be a differentiable quasi-convex function of the n-dimensional vector x, and let $g(x)$ be an m-dimensional differentiable quasi-convex vector function, both defined for x^0. Let x^0 and λ^0 satisfy the Kuhn–Tucker–Lagrange conditions, and let one of the following conditions be satisfied:*

   a) $f_{x_{i_0}} > 0$ for at least one variable x_{i_0} ;

   b) $f_{x_{i_1}} < 0$ for some relevant variable x_{i_1} ;

   c) $f_x \neq 0$ and $f(x)$ is twice differentiable in the neighborhood of x^0;

   d) $f(x)$ is convex.

   then x^0 minimizes $f(x)$ subject to the constraints $g(x) \leq 0,\ x > 0$. If $m^* \geq 1,\ n^* \geq 1$ then (m^*, n^*) satisfies (21) with $\lambda_1 = 0,\ \lambda_2 = 0$. The condition d) is fulfilled. If $m^* \geq 1,\ n^* < 1$ then $(\sqrt{\frac{a_2(W+S)}{R(a_1+a_3)}}, 1)$ satisfies (21) with $\lambda_1 = 0, \lambda_2 > 0$. The condition a) is fulfilled: $\lambda_2(\sqrt{C}, 1) = \frac{\partial L}{\partial n} > 0$. Let $m^* < 1,\ n^* < 1$ and $\lambda_1(1,1) \geq 0,\ \lambda_2(1,1) \geq 0$ then $m = 1,\ n = 1$. The condition c) is fulfilled.

5. Substituting $(\sqrt{A}, \sqrt{B})$ into $L(m,n)$ we obtain:

$$L(\sqrt{A}, \sqrt{B}) = \sqrt{2d}(\sqrt{Wa_1} + \sqrt{Ra_2} + \sqrt{Sa_3}) = L_2$$

   and in the same way

$$L(1,1) = \sqrt{2d(W+R+S)(a_1+a_2+a_3)} = L_3$$
$$L(\sqrt{C}, 1) = \sqrt{2d}(\sqrt{(W+S)(a_1+a_3)} + \sqrt{Ra_2}) = L_1$$
$$L(1, \sqrt{D}) = \sqrt{2d}(\sqrt{(W+R)(a_1+a_2)} + \sqrt{Sa_3}) = L_4$$

□

Appendix B

Property 1. *The functions $A(\alpha), C(\alpha)$ are positive and decreasing; if $\alpha \in (0,1)$, the functions $B(\alpha), D(\alpha)$ are positive and increasing, if $\alpha \in (0,1)$.*

Proof. The functions $A(\alpha)$ and $B(\alpha)$ are obviously positive at any $\alpha \in (0,1)$. The function $A(\alpha)$ is decreasing on $(0,1)$, if for any $\alpha \in (0,1)$: $A'(\alpha) < 0$. The latest is true:

$$A'(\alpha) = -\frac{W(h+u)}{ru}\frac{1}{\alpha^2} < 0, \quad \alpha \in (0,1).$$

It can be shown in the same way that $B'(\alpha) > 0$, if $\alpha \in (0,1)$. Hence $B(\alpha)$ increases on $(0,1)$. To prove that $C(\alpha)$ is positive on $(0,1)$ consider the inequality $C(\alpha) > 0$. Let $u > h$ then $\frac{u}{u-h} > 1$. Then $C(\alpha) > 0$, if $\alpha \in (0,1) \cup (1, \frac{u}{u-h})$. Let $u < h$ then $\frac{u}{u-h} < 0$. Then $C(\alpha) > 0$, if $\alpha \in (-\infty, \frac{u}{u-h}) \cup (0,1) \cup (0, +\infty)$. In both cases $C(\alpha) > 0$ on $(0,1)$. To prove $C(\alpha)$ decreases, consider the inequality $C'(\alpha) < 0$ we obtain:

$$\frac{(W+s)(h+u)}{r} \frac{(\alpha-1)(u+\alpha(2h-u))}{\alpha^2(u+\alpha(h-u))^2} < 0.$$

There are three cases: if $u > 2h$ then $\alpha \in (-\infty, 0) \cup (0,1) \cup (\frac{u}{u-2h}, \frac{u}{u-h}) \cup (\frac{u}{u-h}, +\infty)$, if $2h > u > h$ then $\alpha \in (\frac{u}{u-2h}, 0) \cup (0,1)$, if $h > u$ then $\alpha \in (\frac{u}{u-2h}, 0) \cup (0,1)$. In any case $C'(\alpha) < 0$ on $(0,1)$. This means that $C(\alpha)$ decreases on $(0,1)$. It can be shown in the same way that $D(\alpha)$ is positive and the derivative

$$D'(\alpha) = \frac{(W+r)h}{s} \frac{\alpha(2(u+h) - \alpha(u+2h))}{(1-\alpha)^2(u+h-\alpha h)^2} > 0$$

at any $\alpha \in (0,1)$. Hence $D(\alpha)$ increases on $(0,1)$. □

Property 2. *Each of the equations $A(\alpha) = 1,\ B(\alpha) = 1,\ C(\alpha) = 1,\ D(\alpha) = 1$ has a unique solution $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, correspondingly, if $\alpha \in (0,1)$:*

$$\alpha_1 = \frac{W(h+u)}{ru + W(h+u)},$$
$$\alpha_2 = \frac{su}{su + Wh},$$
$$\alpha_3 = \begin{cases} \frac{1}{2}\frac{2(W+s)(h+u)+ru-\sqrt{4hr(W+s)(h+u)+r^2u^2}}{(W+s)(h+u)-r(h-u)}, & (W+s)(h+u)-r(h-u) \neq 0 \\ \frac{h-u}{2h+u}, & (W+s)(h+u)-r(h-u) = 0 \end{cases},$$
$$\alpha_4 = \begin{cases} \frac{1}{2}\frac{-s(u+2h)+\sqrt{4hs(W+r)(h+u)+u^2s^2}}{h(W+r-s)}, & W+r-s \neq 0 \\ \frac{u+h}{u+2h}, & W+r-s = 0 \end{cases}.$$

Proof. The proof is simply to prove that α_1 is the unique solution of the equation $A(\alpha) = 1$ and α_2 is the unique solution of the equation $B(\alpha) = 1$. Consider the equation $C(\alpha) = 1$. It is equivalent to:

$$\frac{((W+s)(h+u) - r(h-u))\alpha^2 - (2(W+s)(h+u) + ru)\alpha + (W+s)(h+u))}{r\alpha(u+\alpha(h-u))} = 0.$$

If $(W+s)(h+u) - r(h-u) \neq 0$, the numerator has two roots:

$$\alpha_3^1 = \frac{1}{2}\frac{2(W+s)(h+u) + ru + \sqrt{4hr(W+s)(h+u) + r^2u^2}}{(W+s)(h+u) - r(h-u)},$$
$$\alpha_3^2 = \frac{1}{2}\frac{2(W+s)(h+u) + ru - \sqrt{4hr(W+s)(h+u) + r^2u^2}}{(W+s)(h+u) - r(h-u)}.$$

We simply show that if $(W+s)(h+u)-r(h-u)>0$ then $\alpha_3^1>1$ and if $(W+s)(h+u)-r(h-u)<0$ then $\alpha_3^1<0$. Regardless of whether $(W+s)(h+u)-r(h-u)$ is positive or not, $\alpha_3^2\in(0,1)$. If $(W+s)(h+u)-r(h-u)=0$ then the numerator has the unique solution $\alpha_3^0=\frac{h-u}{2h+u}$. It is obvious that $\alpha_3^0<1$. It is necessary to prove that $\alpha_3^0>0$. From $(W+s)(h+u)-r(h-u)=0$ it follows that $u=\frac{W+s-r}{W+s+r}h<h$. Hence $\alpha_3^0>0$, which is our proof. The denominator has two roots $\alpha=0$ and $\alpha=\frac{u}{u-h}$. Neither root is in interval $(0,1)$. Thus it was proved that the equation $C(\alpha)=1$ has a unique solution on $(0,1)$, which is equal to

$$\alpha_3=\begin{cases}\frac{1}{2}\frac{2(W+s)(h+u)+ru-\sqrt{4hr(W+s)(h+u)+r^2u^2}}{(W+s)(h+u)-r(h-u)}, & (W+s)(h+u)-r(h-u)\neq 0\\ \frac{h-u}{2h+u}, & (W+s)(h+u)-r(h-u)=0\end{cases}.$$

The roots of the equation $D(\alpha)=1$ can be found in the same way. ☐

Property 3. *The functions $A(\alpha)$ and $C(\alpha)$ have two common points, if $\alpha\in(0,1]$: $(1,0)$ and $(\alpha_2,\frac{W^2h(h+u)}{rsu^2})$; moreover: $A(\alpha)\geq C(\alpha)$, if $\alpha\in[\alpha_2,1]$, $A(\alpha)<C(\alpha)$, if $\alpha\in(0,\alpha_2)$. The functions $B(\alpha)$ and $D(\alpha)$ have two common points, if $\alpha\in[0,1)$: $(0,0)$ and $(\alpha_1,\frac{W^2h(h+u)}{rsu^2})$; moreover: $B(\alpha)\geq D(\alpha)$, if $\alpha\in[0,\alpha_1]$, $B(\alpha)<D(\alpha)$ if $\alpha\in(\alpha_1,1)$.*

Proof. According to Property 1, $A(\alpha)$ and $C(\alpha)$ are both decreasing functions. Consider the following inequality:

$$A(\alpha)\geq C(\alpha)\Leftrightarrow\frac{(1-\alpha)(\alpha(us+hW)-us)}{u\alpha(u+\alpha(h-u))}\geq 0.$$

If $u>h$ then $\alpha\in(-\infty,0)\cup[\frac{us}{us+hW},1]\cup(\frac{u}{u-h},+\infty)$. If $u<h$ then $\alpha\in(\frac{u}{u-h},0)\cup[\frac{us}{us+hW},1]$. Hence, if $\alpha\in(0,1)$ then $A(\alpha)\geq C(\alpha)$ on $[\frac{us}{us+hW},1]=[\alpha_2,1]$, which was to be proved. This can be obtained in the same way as $B(\alpha)\geq D(\alpha)$, if $\alpha\in[0,\alpha_1]$. ☐

Property 4. *The equations $A(\alpha)=B(\alpha)$ and $C(\alpha)=D(\alpha)$ have unique solutions α^* and α^{**}, correspondingly. Moreover, if $C(\alpha^{**})>1$ then $A(\alpha^*)>1$, if $C(\alpha^{**})<1$ then $A(\alpha^*)<1$, if $C(\alpha^{**})=1$ then $A(\alpha^*)=1$ and $\alpha^*=\alpha^{**}$, where*

$$\alpha^*=\frac{\sqrt{\frac{(h+u)s}{ru}}}{1+\sqrt{\frac{(h+u)s}{ru}}}$$
$$A(\alpha^*)=B(\alpha^*)=\frac{W}{u}\sqrt{\frac{h(h+u)}{rs}}\tag{23}$$

Proof. Consider the following equation:

$$A(\alpha)=B(\alpha)\tag{24}$$

It has a unique solution since $A(\alpha)$ is positive and monotonously decreasing and $B(\alpha)$ is positive and monotonously increasing for $\alpha \in (0,1)$. Now, let α^* be the rate at which (24) holds. The value of α^* is obviously equal

$$\alpha^* = \frac{\sqrt{\frac{(h+u)s}{ru}}}{1+\sqrt{\frac{(h+u)s}{ru}}}$$

and is positive and less than one. Furthermore,

$$A(\alpha^*) = B(\alpha^*) = \frac{(s_1 - s + r_1 - r)}{u}\sqrt{\frac{h(h+u)}{rs}} = \frac{W}{u}\sqrt{\frac{h(h+u)}{rs}}.$$

Function $C(\alpha)$ is positive and monotonously decreasing for $\alpha \in (0,1)$. Since $C(1) = 0$ and $\lim_{\alpha\to 0} C(\alpha) = +\infty$, then the range of values that the function $C(\alpha)$ can take is an interval $[0,+\infty)$. $D(\alpha)$ is positive and monotonously increasing for $\alpha \in (0,1)$ with the range of values $[0,+\infty)$. This means that equation $C(\alpha) = D(\alpha)$ has a unique solution if $\alpha \in (0,1)$ and curves $C(\alpha)$ and $D(\alpha)$ have one intersection point if $\alpha \in (0,1)$. Recall that $\alpha = \alpha^*$ if (24) holds. Denote by α^{**} the solution of $C(\alpha) = D(\alpha)$, if $\alpha \in (0,1)$. It can be proved that if $C(\alpha^{**}) > 1$ then $A(\alpha^*) > 1$ and if $C(\alpha^{**}) < 1$ then $A(\alpha^*) < 1$. At first we prove that

$$\begin{cases} C \geq 1 \\ D \geq 1 \end{cases} \Rightarrow \begin{cases} A \geq 1 \\ B \geq 1. \end{cases}$$

For example, let $Sa_2 - Ra_3 \geq 0 \Leftrightarrow A > B$ then

$$\begin{cases} C \geq 1 \\ D \geq 1 \\ A > B \end{cases} \Leftrightarrow \begin{cases} (Wa_2 - Ra_1) + \underbrace{(Sa_2 - Ra_3)}_{\geq 0} \geq 0 \\ (Wa_3 - Sa_1) - \underbrace{(Sa_2 - Ra_3)}_{\geq 0} \geq 0 \\ A > B \end{cases} \Rightarrow$$

$$\begin{cases} Wa_3 - Sa_1 > 0 \\ A > B \end{cases} \Rightarrow \begin{cases} B > 1 \\ A > B \end{cases} \Rightarrow A > B > 1.$$

It can be proved that if $Sa_2 - Ra_3 < 0$ then $B > A > 1$. In the same way it can be proved that

$$\begin{cases} C < 1 \\ D < 1 \end{cases} \Rightarrow \begin{cases} A < 1 \\ B < 1. \end{cases}$$

We obtain that if at some $\bar{\alpha} \in (0,1), A(\alpha) > 1$ and $B(\alpha) > 1$ then $A(\alpha^*) = B(\alpha^*) > 1$. If this is not so, for example $A(\bar{\alpha}) > B(\bar{\alpha}) > 1 > A(\alpha^*) = B(\alpha^*), \alpha^* < \bar{\alpha}$ then $A(\alpha)$ and $B(\alpha)$ are both increasing functions, which is not true. Consequently, if $C(\alpha^{**}) = D(\alpha^{**}) > 1$ then $A(\alpha^*) = B(\alpha^*) > 1$ and if $C(\alpha^{**}) = D(\alpha^{**}) < 1$ then $A(\alpha^*) = B(\alpha^*) < 1$. Now to prove that if $C(\alpha^{**}) = 1$ then $A(\alpha^*) = 1$. Let $C(\alpha^{**}) = 1$ then at $\alpha = \alpha^*$:

$$\begin{cases} C = \frac{a_2(S+W)}{R(a_1+a_3)} = 1 \\ D = \frac{a_3(W+R)}{S(a_1+a_2)} = 1. \end{cases} \tag{25}$$

If $a_2 S = a_3 R$ then from (25) it follows that

$$\begin{cases} A = \frac{Wa_2}{Ra_1} = 1 \\ B = \frac{Wa_3}{Sa_1} = 1. \end{cases}$$

If $a_2 S > a_3 R$ then from (25) it follows that

$$\begin{cases} A = \frac{Wa_2}{Ra_1} < 1 \\ B = \frac{Wa_3}{Sa_1} > 1, \end{cases} \tag{26}$$

and from $a_2 S > a_3 R$ it follows that $A > B$, which conflicts with (26). This means that if $\alpha = \alpha^{**}$ then $a_2 S = a_3 R$, $A = B = C = D = 1$ and $\alpha^* = \alpha^{**}$. □

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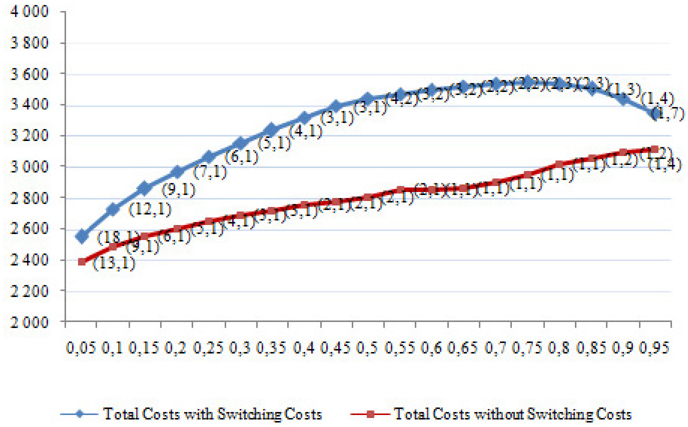

Figure 3: Example 1

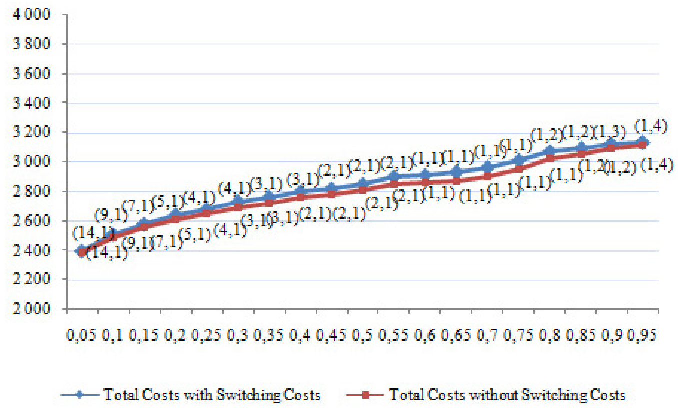

Figure 4: Example 2