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Simultaneous and Sequential Voting under General Decision Rules

Friedel Bolle

Contact:

Friedel Bolle, Europa-Universität Viadrina Frankfurt (Oder) Grosse Scharrnstrasse 59, D - 15230 Frankfurt (Oder), Germany

bolle@europa-uni.de

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Simultaneous and Sequential Voting under General Decision Rules

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Europa-Universität Viadrina Frankfurt (Oder)

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Abstract. In an economic theory of voting, voters have positive or negative costs of voting in favor of a proposal and positive or negative benefits from an accepted proposal. When votes have equal weight then simultaneous voting mostly has a unique pure strategy Nash equilibrium which is independent of benefits. Voting with respect to (arbitrarily small) costs alone, however, often results in voting against the “true majority” (Groseclose and Milyo, 2010). If voting is sequential as in the roll call votes of the US Senate then, in the unique subgame perfect equilibrium, the ”true majority” prevails (Groseclose and Milyo, 2013). It is shown that the result for sequential voting holds also with different weights of voters (shareholders), with multiple necessary majorities (EU decision making), or even more general rules. Simultaneous voting in the general model has more differentiated results.

Keywords: Voting, free riding, binary decisions, unique pure strategy equilibria

JEL: H41, D71

Europa-Universität Viadrina Frankfurt (Oder)
Grosse Scharmstrasse 59
D - 15230 Frankfurt (Oder), Germany
Email: bolle@europa-uni.de

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1. Introduction

The decision rules of the EU changed substantially with the growth of the union. According to the Treaty of Nice, acts by the EU council had to be supported by a minimum number of countries with a minimum number of weighted votes and a minimum total population. In 2017, according to the Treaty of Lisbon and after a three years transition period, the triple majority was reduced to a double majority. Now, a successful vote in the EU Council requires a qualified majority of countries, 55% if acting on a proposal from the Commission or from the High Representative, or else 72%, and, in addition, a qualified majority of population (65%). Till now, there is no general theory of voting which covers such decision rules.

The economic and political relations of countries are more and more governed by membership in multinational organizations instead of bilateral contracts. Decision making in these organizations has to take into account considerable asymmetry concerning the size of the countries as well as their economic, political, and military power. This leads to veto rights, weighted voting, and the requirement of multiple (super-) majorities. Posner and Sykes (2014) provide a survey of the plethora of voting rules in international organizations (GATT/WTO, World Bank, IMF, International Seabed Authority, EU), they highlight unsatisfactory implications of these rules, and they argue that the even less satisfactory, equal weight majority rule survives only in organizations without decision making authority. In business, weighted (super-) majority voting by shareholders is the normal case.

Voting under such rules is quite different from simple majority voting and it seems important to understand its implications. In this paper, I suggest an economic model of voting which implies results of Groseclose and Milyo (2010, 2013) for majority voting with equal weights. The model covers weighted voting as well as requirements of multiple majorities and all kinds of other decision rules based on Yes-No voting. The central questions are, first, whether or not there is a unique pure strategy equilibrium of the voting game and, second, if such an equilibrium exists, whether or not it has the same result as sincere voting. Third, we ask whether Groseclose and Milyo’s (2010,

1 Sincere voting is the optimal vote if a voter is decisive.
2013) characterization of equilibria survives the generalization to more general voting rules. In addition to Groseclose and Milyo (2010, 2013) there is no really close literature. This is shown in more detail in Bolle (2017). As an example, at the end of Section 3, we will briefly discuss Dal Bo (2007) who proposes a model which is otherwise close to mine, differs however in two crucial aspects.

In the next section I present some basic assumptions of the model which are independent of the decision rule. Then, in order to strengthen the intuition for the general case, I (very briefly) derive Groseclose and Milyo’s (2010) result for simultaneous voting. In Section 4, I suggest a model with general decision rules and highlight similarities and differences to simultaneous voting with equal weights. Section 5 shows that sequential voting has the same outcome as sincere voting not only for majority voting (Groseclose and Milyo, 2013) but also under general decision rules. Section 6 is the conclusion.

2. Values in voting and equilibrium selection

We assume that a voting body has to decide on a proposal by Yes-No voting. The threshold for the acceptance of the proposal will be described later.

**Assumption 1:** If the proposal is not accepted and if player \( i \) votes No then her revenue is \( R_i = 0 \), i.e., the status quo is evaluated by 0. Player \( i \) bears costs \( c_i \) if she votes Yes and she enjoys benefits \( G_i \) if the proposal is accepted. Players want to maximize their revenues \( R_i \) which are benefits minus costs.

Costs may be intrinsic\(^2\) (party or personal loyalty, religious requirements, conscience) or extrinsic, for example caused by the threats of party whips or bribes (negative costs) from lobbyists or their equivalents in international organizations, often powerful and/or rich countries. Extrinsic voting costs may also stem from the effect of a vote on public opinion and reputation (more in Bolle, 2017). Benefits evaluate personal advantages

\(^2\) Mainly intrinsic values of voting for or against a proposal or candidate are assumed in the literature on “expressive preferences” (Brennan and Lomasky, 1997; Hillman, 2010), where voters want others to show their preferences by voting in favor of a party and do not follow strategic considerations; but in that literature the “paradox of voting” is of central interest and not the equilibrium of voting on a certain proposal.
and disadvantages as well as, according to a voter’s personal judgement, positive or negative public consequences from a successful vote.

We distinguish four cases. If \( 0 < c_i < G_i \), player \( i \) wants the proposal to be accepted without voting approvingly. If \( c_i < 0 < G_i \) or \( c_i < G_i < 0 \), \( i \) has the dominant strategy to vote Yes. If \( 0 < G_i < c_i \) or \( G_i < 0 < c_i \), \( i \) has the dominant strategy to vote No. If \( G_i < c_i < 0 \), then \( i \) wants to free-ride on the No votes of others.

**Lemma 1:**

(i) Voters with dominant strategies vote according to their costs of voting. Voters with negative costs vote Yes, voters with positive costs vote No.

(ii) If voters with dominant strategies determine the outcome of the vote, then all players vote according to their costs.

(Without proof)

In the following we assume that voters with dominant strategies do not decide the vote alone. We neglect them all after taking their decisions into account. In equal weight majority voting this means that a minimum number \( k \) of players have to vote Yes where \( k \) can be any number between 1 and the number of remaining voters.

**Definition 1:** \( N = N^+ + N^- \) is the set of voters. For all \( i \in N^+ \) we have \( 0 < c_i < G_i \) and for all \( i \in N^- \) we have \( G_i < c_i < 0 \). The number of voters in \( N^+ \) and \( N^- \) are \( n^+ \) and \( n^- \).

Pure strategy equilibria in voting games (general definition below) are not only “simpler” than equilibria with mixed strategies, they also need only qualitative information, namely common knowledge about the set \( N^+ \) or \( N^- \) to which each player belongs. A severe problem is coordination on one of the equilibria if there are many.

**Assumption 2:** If there is a unique pure strategy equilibrium, it is played by the voters.

### 3. Simultaneous equal weight voting

Imagine the members of a parliament simultaneously voting on a parliamentary pay rise. Acceptance of the proposal requires \( k \) Yes votes from the \( n \) members without a
dominant strategy. All n remaining members want the vote to be successful, but no one wants to be blamed for a positive vote (i.e., all are from $N^+$). Then, there are $\binom{n}{k}$ pure strategy equilibria where a minimal winning coalition of k voters vote Yes and the others vote No. For k>1, there is one additional pure strategy (free rider) equilibrium where all vote No.

Now let us assume that one of the members has contrary preferences. Member m is from $N^-$; she does not want a parliamentary pay rise but, because of solidarity with her fellow members, neither does she want to vote against it. If and only if her vote is decisive, she would vote No. This seemingly unimportant change of the composition of voters has far-reaching consequences. All the pure strategy equilibria with minimal winning coalitions are destabilized. $m$ cannot be a member of the coalition, because then her vote is decisive. Otherwise she would join every winning coalition with the consequence that all other members have an incentive to leave it. For k>2, the only equilibrium which remains is the free rider equilibrium, where member m votes Yes in order to incur the negative costs and all other members vote No in order to avoid the positive costs.

The same rationale applies if we allow $n^-$ to be an arbitrary number. Let $S^*$ be the equilibrium set of Yes voters. If $|S^*| \neq k$ then no vote from $S^*$ is decisive; therefore no voter from $N^+$ and every voter from $N^-$ is in $S^*$. If $|S^*| \neq k - 1$ then no voter from $N - S^*$ is decisive; all voters from $N^-$ want to join $S^*$ and no voter from $N^+$ wants to join. Therefore, with the exception of $n^- = k$ or $k - 1$, there is a unique equilibrium set $S^* = N^-$. $|S^*| = k$ would imply that all voters from $N^-$ who are in $S^*$ want to leave it and all outside $S^*$ want to join it; therefore such an equilibrium does not exist. The same applies for $|S^*| = k - 1$ where all voters from $N^+$ who are in $S^*$ want to leave it and all outside $S^*$ want to join it.

**Proposition 1** (Groseclose and Milyo, 2010): If $0 < n^- < n$ and $n^- \neq k, k - 1$ then there is a unique pure strategy (free rider) equilibrium where all players decide according to their costs.

**Definition 2:** A proposal is said to have a **true majority** if, for equal weight voting, $|N^+| = n^+ \geq k$, or, more general, if a proposal is accepted when the voters from $N^+$ vote Yes.
Corollary 1: If $0 < n^- < n$ and $n^- \neq k, k - 1$, then the result of equilibrium play is contrary to the true majority decision if and only if $n^- < k \leq n^+$ or $n^+ < k \leq n^-$, i.e., if $k$ is between $n^+$ and $n^-$. 

Proof: Proposition 1.

This discussion has shown that there is an eminent difference between cases where all voters have qualitatively equal preferences ($n^+ = 0$ or $n^- = 0$) and other cases. This has an interesting consequence for bribing (when bribes depend only on the vote). For the sake of simplicity assume that all voter have costs and benefits $G_i < 0 < c_i$ or $0 < G_i < c_i$, i.e., all have dominant strategies of voting No. Bribing for Yes votes reduces $c_i$ with the consequence that some voters have dominant strategies of voting Yes and some belong to $N^+$ and $N^-$. From the viewpoint of a bribing agent who wants the proposal to pass, bribing should not result in a situation with $n^+ = 0$ and $k > 0$ because he creates a game with many pure strategy equilibria. It seems most plausible that, in such a situation, a mixed strategy equilibrium is played. Therefore, either bribes must be so large that enough dominant strategy voters vote Yes ($k=0$) or be low enough to leave some voters in $N^-$. In the latter case, only the free rider equilibrium survives.

There is extensive literature on bribing. Dal Bo (2007) proposes a model close to the one above, however with $n^- = 0$ or $n^+ = 0$. He allows a lobbyist to offer a “tricky” bribing contract, namely paying a very high price for a vote which turns out to be decisive and paying very little otherwise. Thus the lobbyist can buy, for almost nothing, a majority even if all voters have opposite preferences. Dal Bo (2007) discusses consequences from his result and also a lot of variations of assumptions; but all this does not touch results from my model which concentrates on $n^- > 0, n^+ > 0$ and more general decision rules (below). The same applies to other voting models from the literature (see Bolle, 2017).

4. General voting games

Definition 3: In a voting game, there are $n \geq 2$ players who simultaneously or sequentially vote "Yes" or “No” on a certain proposal. $N = \{1, ..., n\}$ is the player set.
\(\mathcal{H}\) designates the set of all subsets of \(N\) whose Yes votes suffice to accept the proposal. It has the following properties: The empty set \(\emptyset \notin \mathcal{H}\); \(N \in \mathcal{H}\); if \(S \subset S' \subset N\) and \(S \in \mathcal{H}\) then also \(S' \in \mathcal{H}\).

(i) We call \(S \in \mathcal{H}\) a minimal supporting set if no strict subset of \(S\) is contained in \(\mathcal{H}\).

(ii) Player \(i\) is called a pivot player with respect to \(S\) if \(S \cup \{i\} \in \mathcal{H}\) and \(S - \{i\} \notin \mathcal{H}\).

(iii) If \(\mathcal{H} = \{S; |S| \geq k\}\), we call the game an equal weight voting game.

Definition 3 is similar to the definition of Simple Cooperative Games where the characteristic function takes binary values. We will establish, however, a non-cooperative game with more structure.

Assumption 3: We assume that \(N^-\), \(N^+\) and \(\mathcal{H}\) are common knowledge.

Note that the voters need not know the exact values of \(G_i\) and \(c_i\). Let us first characterize simultaneous voting games with unilateral interests.

Proposition 2: Let us assume \(n^- = 0\).

(i) There are as many pure strategy equilibria with the acceptance of the proposal as there are minimal supporting sets. All \(i \in S \in \mathcal{H}\), with \(S = \) minimal set, vote Yes in such an equilibrium and all other players vote No.

(ii) If \(\{i\} \notin \mathcal{H}\) for all \(i\), then voting No by all \(i\) is the only pure strategy equilibrium without the acceptance of the proposal; otherwise no such equilibrium exists.

Proof: In both cases no player can gain from changing her decision. In an equilibrium without the acceptance of the proposal no player would incur costs by voting Yes. □

Respective equilibria exist for \(n^+ = 0\). With many different minimal supporting sets, coordination of the players is extremely difficult and the question arises whether completely mixed strategy equilibria (in particular when they are unique or Pareto-ranked) are more plausible candidates for equilibrium selection.

Proposition 3 (necessary conditions for pure strategy equilibria): Let \(S^* \subset N\) denote the set of players who vote Yes in a pure strategy equilibrium.
(i) If the proposal is accepted then \( S^* \in \mathcal{H}, N^- \subseteq S^*, S^* - \{i\} \in \mathcal{H} \) for \( i \in N^- \), and \( S^* - \{j\} \notin \mathcal{H} \) for \( j \in S^* - N^- \).

(ii) If the proposal is rejected then \( S^* \notin \mathcal{H}, S^* \subset N^- \), \( \{j\} \cup S^* \notin \mathcal{H} \) for \( j \in N^+ \), and \( \{i\} \cup S^* \in \mathcal{H} \) for all \( i \in N^- - S^* \).

**Proof:** (i) When the proposal is accepted, every player \( i \in N^- - S^* \) is, because of her negative costs, better off if she votes Yes, i.e., \( N^- \subset S^* \). Every player \( i \in N^- \) would, however, withdraw her support if she were a pivot player; \( S^* - \{i\} \in \mathcal{H} \) expresses that she is not. Every player \( j \in S^+ - N^- \) would, because of her positive costs, withdraw her support if she is not a pivot player; \( S^+ - \{j\} \notin \mathcal{H} \) expresses that she is.

(ii) In a pure strategy equilibrium without the acceptance of the proposal, support comes from \( S^* \notin \mathcal{H} \). No player from \( N^+ \), because of her positive costs, would support a rejected proposal, i.e., \( S^* \subset N^- \), except she were a pivot player; therefore \( \{j\} \cup S^* \notin \mathcal{H} \) for \( j \in N^+ \) is necessary. All other players from \( N^- \) do not vote Yes because otherwise the proposal would be accepted, i.e., \( \{i\} \cup S^* \in \mathcal{H} \) is required for all \( i \in N^- - S^* \). □

All players have incentives to free ride which means that players from \( N^- \) vote Yes and players from \( N^+ \) vote No. They deviate from this behavior only if they are pivot players. Therefore the set of Yes-voters is at least \( N^- \) in (i) and the set of No-voters is at least \( N^+ \) in (ii). In the following we characterize cases where the result of majority voting (with \( n^+ = k, k - 1 \) ) applies also in the general case, namely \( N^- \) being the only equilibrium set of Yes-voters.

**Definition 4:** Player \( j \) is said to be replaceable\(^3\) by \( i \) if, for every \( S \subseteq N - \{i, j\} \) \( S \notin \mathcal{H} \), and \( S \cup \{j\} \in \mathcal{H} \), also \( S \cup \{i\} \in \mathcal{H} \) applies. \( i \) and \( j \) are said to be mutually replaceable if \( i \) is replaceable by \( j \) and \( j \) is replaceable by \( i \).

**Lemma 2:** If all pairs of players are mutually replaceable, then the game is representable as an equal weight BTPG game, i.e., the threshold structure is \( \mathcal{H} = \{S: |S| \geq k\} \) with \( 1 \leq k \leq n \).

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\(^3\) Isbell (1958) uses the same definition but says that “\( i \) is as least as desirable as \( j \)”. In the literature on power indices desirability is used to characterize local monotonicity, i.e., if \( i \) is as least as desirable as \( j \), then \( i \)'s power index is not lower than \( j \)'s index (Freixas and Gambarelli 1997).
Proof: If \( S \in \mathcal{H} \) is a minimal supporting set with the lowest number \( k \) of members, then every player can be substituted by a player from \( N - S \). I.e., if a set with \( k \) players is in \( \mathcal{H} \), then every set with at least \( k \) players is in \( \mathcal{H} \). As \( k \) has been defined as the minimal number of members of a supporting set no other sets are in \( \mathcal{H} \). □

In general voting games, there may be no replaceability relations between pairs of players or they may be one-sided or mutual. Examples of one-sided replaceability are shareholders with sufficiently unequal shares. The following is an example without replaceability relations.

Example 1: There are four countries (1, 2, 3, 4) with weights (2, 4, 1, 5) and populations (40, 20, 50, 10). Acceptance of a proposal requires aggregate weights of Yes-voters of at least 6 and an aggregate population of at least 60.

In this example the only minimal supportive sets of countries are \{1,2\} and \{3,4\}. Therefore, no player is replaceable by another player. We will come back to this example later on.

Proposition 4 (sufficient conditions for \( S^* = N^- \)):

(i) If \( N^- - \{i\} \in \mathcal{H} \) for all \( i \in N^- \) then \( S^* = N^- \) describes the unique pure strategy equilibrium with the acceptance of the proposal. If, in addition, every \( j \in N^- \) is replaceable by a player \( i(j) \in N^+ \), then no pure strategy equilibrium without the acceptance of the proposal exists.

(ii) If \( N^- \cup \{i\} \notin \mathcal{H} \) for all \( i \in N^+ \), then \( S^* = N^- \) describes the unique pure strategy equilibrium without the acceptance of the proposal. If, in addition, every \( i \in N^+ \) is replaceable by a player \( i(j) \in N^- \), then no pure strategy equilibrium with the acceptance of the proposal exists.

Proof: (i) \( S^* = N^- \) fulfills the necessary conditions from Proposition 3 (i). In addition, no player from \( N^+ \) or \( N^- \) has an incentive to deviate from \( S^* = N^- \) which therefore describes the unique equilibrium. In an equilibrium with the rejection of the proposal, according to Proposition 3 (ii), \( S^* \) contains only players from \( N^- \) but not all (because \( N^- \in \mathcal{H} \)) and all players outside \( S^* \) are pivot players. But as each of these pivot players can be replaced by a player from \( N^+ \) the necessary conditions from Proposition 3 (ii) do not apply. (ii) According to Proposition 3 (ii), \( S^* = N^- \) describes the unique
equilibrium with the rejection of the proposal. Because of \( N^- \notin \mathcal{H} \) an equilibrium \( S^* \) with the acceptance of the proposal has to contain additional (pivot) players from \( N^+ \) and, therefore, also some of the players from \( N^- \) are pivot players and would vote No. \( \square \)

Proposition 1 is implied by Propositions 4. Also for more general cases, the conditions of Proposition 4 can easily be checked.

**Definition 4:** In a weighted voting game, every voter has a weight (a positive real number) and a proposal is accepted if and only if the Yes voters have an aggregate weight larger than or equal to a threshold \( k \).

The players with maximal weights \( i_+ (i_-) \) from \( N^+ (N^-) \) play a special role. In the general voting game their equivalents are players \( i_+ (i_-) \) from \( N^+ (N^-) \) who can replace all other players in their set. In general voting games such players need not exist.

**Corollary 2:** If there are players \( i_+ (i_-) \) who can replace all other players in \( N^+ (N^-) \) then:

(i) If \( N^- \setminus \{i_-\} \in \mathcal{H} \) then \( S^* = N^- \) describes the unique pure strategy equilibrium with the acceptance of the proposal. If, in addition, \( i_- \) is replaceable by a voter from \( N^+ \), then no pure strategy equilibrium without the acceptance of the proposal exists.

(ii) If \( N^- \cup \{i_+\} \notin \mathcal{H} \) then \( S^* = N^- \) describes the unique pure strategy equilibrium without the acceptance of the proposal. If, in addition, \( i_+ \) is replaceable by a voter from \( N^- \), then no pure strategy equilibrium with the acceptance of the proposal exists.

**Proof:** Proposition 4.

Note that Corollary 2 leaves open the possibility that, contrary to Proposition 1, there are equilibria where voters from \( N^+ \) vote Yes. As an example take a voting game with equal weights of 1 with the exception of the chairman \( m \) who has a weight of 2 (tie-breaking power). If the chairman is from \( N^+ \) and if there are \( k-1 \) voters in \( N^- \), then \( S^* = N^- \cup \{m\} \) is the unique set of Yes voters. With increasing differences of the weights of voters we get more such equilibria. Games with Veto players (e.g., in the
UN Security Council can also be modeled as weighted voting games, but we have to give the veto players extremely large weights compared with ordinary voters (and adapt also the threshold k). Sufficient conditions for the representability as a weighted voting game can be found in Freixas et al. (2017). But not all voting games can be represented as weighted voting games. Example 1 above shows that this conjecture is not true. The sum of weights of voters 1 and 2 as well as those of 3 and 4 have to be equal to or larger than a hypothetical threshold k; therefore, the sum of all weights has to be at least 2k. On the other hand, the sums of weights of voters 1 and 3 as well as 2 and 4 have to be smaller than k; therefore, the sum of all weights must be smaller than 2k.

Example 1 has been derived from the requirement of a double weighted majority. The extant examples of double majorities, however, mostly show a certain regularity which might facilitate the representation as a weighted voting game: In one dimension, the players have the same weight. In a second dimension they have different weights. This applies for current EU decision making, for voting on the Kyoto protocol, and for voting in assemblies of condominium owners (when voting with double majorities concerning owners and apartments is required). This structure provides us with an ordering of the players according to the second dimension which constitutes also an ordering with respect to replaceability, Thus voters and from Corollary 2 exist. But that does imply representability as a weighted voting game. For a counter-example we need the following plausible conjecture.

**Lemma 2:** If, in a weighted voting game, there is a class \( R \subset N \) of mutually replaceable players then there is a representation of the game with weights \( x_i = x_R \) for all \( i \in R \).

**Proof:** Bolle (2017).

**Example 2:** \( n=15 \), \( w_1 = w_2 = \cdots = w_{12} = 1 \), \( w_{13} = w_{14} = w_{15} = 10 \). Acceptance of a proposal requires a majority of 8 players with aggregate weights of 17.

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4 In the UN Security Council there are five permanent and 10 non-permanent members. The council accepts a proposal with a supermajority of 9 votes, which must include all permanent members.

5 Article 25 of the Protocol specifies that the Protocol enters into force when at least 55 countries which account for at least 55% of the total carbon dioxide emissions for 1990 “have deposited their instruments of ratification, acceptance, approval or accession”.


If there is a representation of this game as a weighted voting game with weights $x$, then there is a representation where the twelve “small” players all have a weight $x_1$ and the three “large” players have a weight $x_{15}$: the threshold is $k$. As $\{1, ..., 12\} \notin H$, $x_1 < k/12$. As $\{1, ..., 7, 15\} \in H$, $7x_1 + x_{15} \geq k$. Therefore $x_{15} > 5k/12$. But as $\{13, 14, 15\} \notin H$, $x_{15} < k/3$. Therefore, this example has no representation as a weighted voting game.

4 Sequential voting

In many parliaments, sequential votes are possible or even the normal case. In a roll call vote, the members of the US Senate are required, in alphabetic order, to vote either “yea” or “nay”. Abstention is possible in principle but usually not applied. One may object that, facing a fast sequence of 100 or almost 100 votes, senators have decided on their vote in advance. Nonetheless, let us derive the subgame perfect equilibrium of the sequential voting game.

Let us assume that the order of voters is (voter 1, voter 2, ..., voter n). Again we disregard all voters with dominant strategies after taking their decisions into account so that the sets of remaining necessary votes for the passing of the proposal is again described by $H$. The game of the remaining players consists of a sequence of subgames which are essentially described by

$$S_i^* = (S^* \cap \{1, 2, ..., i - 1\}) \cup (\{i + 1, ..., n\} \cap N^+)$$

where $S^*$ is defined as the set of players who vote Yes. Player i knows $S_i^*$ because she knows who has voted Yes, namely $S^* \cap \{1, 2, ..., i - 1\}$ and because she knows who of the remaining players wants the proposal to be accepted, namely $\{i + 1, ..., n\} \cap N^+$. If $S_i^* \cup \{i\} \in H$, then player i and the remaining players in $N^+$ can enforce the acceptance of the proposal, if $S_i^* \notin H$, player i and the remaining players in $N^-$ can enforce rejection; but $i \in N^+(N^-)$ votes Yes (No) only if she is a pivot player.

**Proposition 5:** The sequential voting game has a unique equilibrium $S^*$.

(i) $i \in N^-$ votes No if she is a pivot player with respect to $S_i^*$; otherwise $i \in N^-$ votes Yes.
(ii) $i \in N^+$ votes Yes if she is a pivot player with respect to $S_i^*$; otherwise $i \in N^+$ votes No.

(iii) If $N^+ \notin \mathcal{H}$ then $S^* \subset N^-$ and the proposal is rejected.

(iv) If $N^+ \in \mathcal{H}$ then $N^- \subset S^*$ and the proposal is accepted.

**Proof:** The proof is by backward induction. Apparently, player $n$ will stick to the rules (i) and (ii). Then the proposal will be accepted if and only if $S_{n+1}^* \in \mathcal{H}$. Let us now assume that the proposal will be accepted if and only if $S_{i+1}^* \in \mathcal{H}$. Then player $i \in N^+$ will induce $S_{i+1}^* \in \mathcal{H}$ if she can. If she is pivotal with respect to $S_i^*$, she must vote Yes, otherwise she saves costs and votes No. If $i \in N^-$ is pivotal with respect to $S_i^*$, she must vote No, otherwise she incurs negative costs and votes Yes. So, (i) and (ii) apply also for player $i$. (iii) and (iv) follow from (i) and (ii). □

For the case of equal weight games, Proposition 4 has been proved by Groseclose and Milyo (2013). A consequence of Proposition 4 is that, for the acceptance of the proposal, the order of voters is irrelevant. Individual votes, however, depend crucially on the order. As an example, take an equal weight voting game with $k = n^+ = n^-$. If the first $k$ players are from $N^+$, all voters vote “Yes”, the first $k$ because they must vote Yes in order to guarantee the acceptance of the proposal, the second $k$ in order to incur the negative costs of voting. If the first $k$ voters are from $N^-$, they vote “Yes” because they cannot prevent the acceptance of the proposal, the second $k$ vote “No” because they need not incur the positive costs of voting “Yes”.

5 Conclusion

An economic model of voting is suggested which assumes Yes-No voting and arbitrary decision rules based on votes. Three crucial assumptions are that there are conflicting interests ($n^+ > 0$, $n^- > 0$), that abstentions are not allowed or not effective, and that there is complete information about qualitative interests and the decision rule ($N^+$, $N^-$, and $\mathcal{H}$ are common knowledge). I think these assumptions are relatively weak. Groseclose and Milyo (2010, 2013) have investigated the case of voters with equal weights, $\mathcal{H} = \{S: |S| \geq k\}$. This paper assumes a general threshold structure $\mathcal{H}$ (Definition 3).
The sequential and the simultaneous voting game have completely different results. In the *sequential* game, the subgame perfect equilibrium is unique. Even with a general threshold structure $\mathcal{H}$, the strict condition for the acceptance of a proposal is that it has a true majority, i.e., $N^+ \in \mathcal{H}$. Who really votes Yes, however, depends on the order of votes.

Under *simultaneous voting with equal weights and a large number of voters*, unique pure strategy equilibria not always but mostly exist. In equilibrium, voters vote Yes if and only if their costs of voting are negative (Groseclose and Milyo, 2010). Therefore, in parliamentary votes, party whips easily succeed. In the general case, Proposition 4 shows, that voting according to costs is a prominent candidate for an (possibly unique) pure strategy equilibrium. But with unequal weights (shareholder decisions) or even multiple necessary majorities (EU decision making), pure strategy equilibria of simultaneous voting are often non-existent or non-unique or such that not all voters vote according to their costs.

**References**


